**Question 5. No Sugar in this Coat**

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• Let Vd ⊆ V be the set of vertices that are at a distance equal to d from s in G, then ∀ i ≥ 0 :  
 u ∈ Vi , v ∈ Vi+1 ⇒ (u, v) ∈ E

This implies that if we start BFS from s, then each node at a level L will be linked to all the nodes at level L+1.

1. **An O(|V |+|E|) time algorithm to find a vertex t ∈ V , such that the following property holds for every vertex u ∈ V : min(dist(u, s), dist(u, t)) ≤ k**

It is obvious that the last nodes (nodes which has maximum) will have the maximum distance from s. Now this last node say M, has the highest tendency to have dist(M,s) >k, we wish to find t such that it satisfies the condition with M just in range, i.e. dist(M,t)=k

Idea is to do BFS once and find the farthest level nodes. Once found we just have to move to its parent k-1 times to find t.

In this figure arrows are just for direction of BFS  
*use the following algorithm to find last node M:*

BFS\_TO\_FIND\_M

->initiate empty queue Q

->push s

-> iniate empty boolean visited[n]=false

-> make visited[s]=true

-> iniate empty array parent[n]=-1 random nonachievable value

while(Q is not empty){

    M=Q.top

    Q.pop

    for(all neighbours v of M){

        if(visited[v]==false){

            visited[v]=true

            parent[v]=M

        }

    }

}

Now, to find t we just need to move up to its parent k-1 times

t=M

for(i=1 to k-1){

    t=parent[t]

}

1. Proof of correctness for the algorithm:

If we consider a graph where each node is assigned to a level based on its distance,this graph has at 3k layers selecting a node from a level no greater than 2k+1 will ensure that the minimum distance between this selected node and both s and t is less than or equal, to k**.**

Dist(t,M)= k, dist(s,M)=?

Upper limit of dist(s,M)=n=3k (given)

Dist(s,M)max=3k

Therefore dist(s,t)max=2k, So any element lying in between will have min(dist(u,s),dist(u,t))<=k

**Assertion :** min(dist(s,u), dist(t,u)) <=k

Proof:

Let the Assertion is false i.e. there exist a u such that min(dist(s,u), dist(t,u)) > k

All the nodes below t are at max distance =k, hence this u don’t lie below t

Nodes at the same level as t are at distance =2 from t, here arises 2 case:

1. k>=2 : here also u cant exist
2. k<2 or k=1 if k=1 total nodes= 3, t=s dist(u,s) for both neighbours of s=1 hence u cant exist here also

u can only exist in middle of s and t,

there are minimum k nodes below t (dist(t,M)=k)

Maximum no. of nodes between s and t = 2k-2

If we take the median elements at distance distance from top = k

And distance from t =k

Hence u cant lie here also

Since U cant lie anywhere in the graph our assumption was wrog,

Contradiction

Hence proved.